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New classes of similarity solutions of the inhomogeneous nonlinear diffusion equations

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Abstract. The Lie similarity method has been used to extend the similarity solutions of the one-dimensional inhomogeneous nonlinear diffusion equations. We determine the Lie point symmetry vector fields and calculate the similarity ansatz. Then we discuss the resulting nonlinear ordinary differential equations. Exact solutions are found, and their relations to some real physical are discussed.

1. Introduction

This paper is concerned with some enlargements of the similarity reductions of the inhomogeneous nonlinear diffusion INLD equation

$$x^p \frac{\partial u}{\partial t}(x, t) = \frac{\partial}{\partial x}(x^m u^n u_x) \tag{1}$$

which is of considerable importance both in physics and mathematics. Also, its special cases have been used successfully to model physical situations in a wide range of fields involving diffusion processes [1–6]. Self-similarity methods were developed [7, 8] for determining similarity reductions of equation (1) and have many applications [9, 10]. Other results using the similarity transformation are given in [11].

In the following we use the method of group-invariant solutions [3–5] to determine new similarity reductions of equation (1), in addition to the previously known ones [9, 7], and to give the group theoretical explanations of the results obtained in [7].

First we determine the Lie point symmetry vector fields [3, 5]. Let

$$U = a(x, t, u)\frac{\partial}{\partial x} + b(x, t, u)\frac{\partial}{\partial t} + c(x, t, u)\frac{\partial}{\partial u} \tag{2}$$

where a, b and c are as yet unspecified functions of x, t and u. We apply the algorithm that provides the symmetry algebra by constructing the prolongation of the vector field U, i.e. the differential operator of the form

$$\mathrm{Pr}^2 U = U + c^x\,\partial_{u_x} + c^t\,\partial_{u_t} + c^{xx}\,\partial_{u_{xx}} \tag{3}$$

where the functions c^t, c^x and c^{xx} are expressed in terms of a, b, c and their derivatives. The prolongation is then applied to equation (1), and the resulting expression is required to vanish on solution of equation (1). This leads to a set of determining equations that must then be solved. Solving equation (3) for a, b and c one obtains the vector field (2) in an explicit form. We find three symmetry vector fields, namely

$$
\begin{aligned}
B_1 &= \partial_t \\
B_2 &= -\frac{n}{v} x\, \partial_x - u\, \partial_u \\
B_3 &= \frac{2}{v} x\, \partial_x + 2t\, \partial_t
\end{aligned}
\tag{4}
$$

where $v=p-m+2$. The symmetry vector fields form a Lie algebra; therefore, $[B_1, B_3]=2B_1$, and all other commutations

$$[B_i, B_j]=[B_i, B_i]=0 \qquad i,j=1,2,3.$$

Because a linear combination of the vector fields determines the general symmetry, we can use a combination of them to classify the types of solutions. By using the adjoint algebra [4], we are able to distinguish four different types of solution corresponding to the basic fields of an optimal system given by B_3, B_2, (B_1+B_2) and (B_2+B_3).

In the following, we demonstrate that these combinations of symmetries produce essential types of solution. We mention that one obtains further solutions of equation (1) by applying finite group transformations to these solutions [3, 4]. Through the characteristic equations.

$$\mathrm{d}x/a=\mathrm{d}t/b=\mathrm{d}u/c$$

we obtain the similarity variable s and the similarity solution $F(s)$ which reduce equation (1) to nonlinear ordinary differential equations, where some of them are solved and the others solved analytically in a restricted region of the (p, m) parameter space.

2. Group-invariant solutions

(a) In order to obtain the group-invariant solutions (similarity solutions) let us first consider the combination of B_2 and B_3 by $w=B_3+kB_2$. We introduce here the parameter k to demonstrate that not only will B_3+B_2 give a specific similarity solution but also a linear combination of B_2 and B_3 with arbitrary coefficients. We choose this combination of vector fields first for clarity of representation. The corresponding finite transformation to w reads

$$\bar{x}=x\,\mathrm{e}^{\varepsilon/r} \qquad \bar{t}=t\,\mathrm{e}^{2\varepsilon} \qquad \bar{u}=u\,\mathrm{e}^{-k\varepsilon}$$

where $r=v/(2-nk)$ and ε is the group parameter.

The group-invariant combinations suggest the similarity variable s and the similarity solution $F(s)$ to be

$$s=xt^{-1/2r} \qquad u(x, t)=t^{-k/2}F(s). \tag{5}$$

Substitution of the similarity into equation (1) results in

$$-s^p\left(\frac{k}{2}F+\frac{s}{2r}F'\right)=\frac{\mathrm{d}}{\mathrm{d}s}(s^m F^n F'). \tag{6}$$

For $k=(p+1)/r$, equation (6) will reduce to the ordinary differential equation

$$-1/2r\frac{d}{ds}(s^{p+1}F)=\frac{d}{ds}(s^{m}F^{n}F') \tag{7}$$

where primes denote differentiation with respect to s. We may integrate equation (7) once to get

$$-1/2r(s^{p+1}F)=s^{m}F^{n}F'+c_1 \tag{8}$$

where c_1 is an arbitrary constant.

For $c_1=0$, further integration of equation (8) gives:

(i) For $n\neq 0$

$$F^{n}=c_2+(n/2rv)s^{v} \qquad v=p-m+2\neq 0 \tag{9}$$

and

$$F^{n}=c_2-(n/2rv)\ln s \qquad v=p-m+2=0. \tag{10}$$

The range of s is determined by requiring the expressions on the right-hand sides to be positive.

(ii) For $n=0$, integration of equation (8) gives

$$F=c_2\,e^{-s^{v}/2rv} \tag{11}$$

where c_2 is an arbitrary constant and $v\neq 0$.

Let us now consider the case $c_1\neq 0$ for some values of n which have some real physical meaning [10], namely $n=-1,\ -\frac{1}{2},\ -2/p+1,\ -2$.

(i) *For* $n=-1$. Equation (8) will be the Bernoulli equation on substitution of $F=1/G$, to give

$$\frac{dG}{ds}-C_1s^{-m}G-(1/2r)s^{p-m+1}=0 \tag{12}$$

which has a solution of the form

$$G=c_2\,e^{c_1s}-\frac{1+sc_1}{2rc_1^2} \tag{13}$$

for $p=m=0$, and the solution

$$G=c_2s^{c_1}+s^2/2r(2-c_1) \tag{14}$$

for $p=m=1$, $c_1\neq 2$.

(ii) *For* $n=-\frac{1}{2}$. The substitution $F=G^2$ transforms equation (8) into the Riccati equation,

$$\frac{dG}{ds}+1/2c_1s^{-m}+1/4rs^{p-m+1}G^2=0 \tag{15}$$

which can be transformed into one of second order by the transformation

$$GH=(4rs^{m-p-1})\frac{dH}{ds}$$

to give

$$sH''+(m-p-1)H'+(c_1/8r)s^{p-2m+2}H=0. \tag{16}$$

Equation (16) can be written in a form of the Bessel equation by substitution of

$$z=\frac{2(c_1/8r)^{1/2}}{p-2m+3}s^{(p-2m+3)/2} \qquad H=S^{(2-m+P)/2}Y(z)$$

to give

$$z^2Y''+zY'+\left[z^2-\left(\frac{p-m+2}{p-2m+3}\right)^2\right]Y=0. \tag{17}$$

Any solution of equation (17) is a cylindrical function, and the general solution is $Y(z)=AX_1(z)+BX_2(z)$ where X_1 and X_2 are two linearly independent cylindrical functions (for details see [12]).

(iii) *For* $n=-2/p+1$. We write $F=s^{-1-p}G(z)$, and $z=\ln s$, then equation (8) reads

$$\frac{dG}{dz}=(1+p)G-(C_1G^{2/p+1}+\frac{1}{2r}G^{1-2/p+1})s^{p-m}. \tag{18}$$

For the case $p=m$, we have an Able equation of the first kind with constant coefficients; and the equation is separable.

(iv) *For* $n=-2$, $p=-m=k$. The substitution $F=2rc_1s^{-k-1}G(s)$ will transform equation (8) to an Able equation of the form

$$\frac{dG}{ds}=2rC_1^2s^{-1}G(A-G-G^2) \tag{19}$$

where $A=(K+1)/2rC_1^2$, which is separable, and an elliptic integral occurs.

(*b*) The similarity reduction is also obtained if we examine the linear combination of B_1 and B_2 by $w=B_1+kB_2$. Here we introduce k in a similar way as above. The finite transformation for this combination can be written as

$$\bar{t}=t+\varepsilon \qquad \bar{x}=x\,e^{-nk\varepsilon/v} \qquad \bar{u}=u\,e^{-k\varepsilon}. \tag{20}$$

If we have the special solution of equation (1) as $u(x, t)$, we obtain from transformation (20) another solution of $\bar{u}(x, t)$, which reads

$$\bar{u}=u(x\,e^{-nk\varepsilon/v}, t+\varepsilon)\,e^{-k\varepsilon}.$$

The general reduction of this subgroup can be obtained by the similarity representation

$$s=x\,e^{nkt/v} \qquad u=e^{-kt}F(s) \tag{21}$$

which reduces equation (1) to

$$-Ks^p\left(F-\frac{n}{v}sF'\right)=\frac{d}{ds}(s^mF^nF'). \tag{22}$$

If we put $n=-v/p+1$ then we may integrate equation (22) once to get

$$F^n = ks^v/(p+1)^2 + C \qquad v \neq 0 \tag{23}$$

where C is an arbitrary constant.

(*c*) A further new class of similarity solutions is generated by the vector field B_2, including only scale invariance with respect to x and u.

The finite transformation for this vector field can be written as

$$\bar{x} = x\,\mathrm{e}^{-n\varepsilon/v} \qquad \bar{t} = t \qquad \bar{u} = u\,\mathrm{e}^{-\varepsilon}.$$

The general reduction of this subgroup can be obtained by the similarity representation

$$s = t \qquad u = x^{v/n} F(s). \tag{24}$$

This solution inserted in equation (1) gives

$$\frac{\mathrm{d}F}{\mathrm{d}s} = \frac{v^2 + nv(1+p)}{n^2} F^{n+1}. \tag{25}$$

Direct integration gives

$$F = \left[C - s\left(\frac{v^2 + vn(p+1)}{n} \right) \right]^{-1/n} \tag{26}$$

where C is an arbitrary constant, $n \neq 0$ and $v = p - m + 2$.

For the case $p = m$, then $v = 2$, and the corresponding solution for u is

$$u(x, t) = x^{2/n} \left[C - \left(\frac{4 + 2n(p+1)}{n} \right) t \right]^{-1/n}. \tag{27}$$

For $n = -2/p + 1$, we get the steady state solution

$$u = A/x^{p+1}.$$

The general solution (27) requires the expression on the right-hand side to be positive, which determines the range of time t.

(*d*) The last vector field of our optimal system which remains to be discussed is B_3, including only scale invariance with respect to x and t. The corresponding new similarity representation is given by

$$s = x^v/t \qquad u = F(s) \tag{28}$$

and the reduced equation of equation (1) reads

$$v^2 s \frac{\mathrm{d}}{\mathrm{d}s}(F^n F') + v(1+p) F^n F' + sF' = 0 \tag{29}$$

which may be solved for some values of p, m and n as follows.

(i) *For* $p = -1$. Equation (29) is integrated once to get

$$v^2 F^n\,\mathrm{d}F/\mathrm{d}s + F = C \tag{30}$$

where C is an arbitrary constant. For $C = 0$, further integration of equation (30) gives for $n < 0$, $m \neq 1$

$$F(s) = (-sn/(1-m)^2)^{1/n}. \tag{31}$$

For $C \neq 0$, let $F = G^{-1}$, then we get for $m \neq 1$

$$v^2\,\mathrm{d}G/\mathrm{d}s = G^{n+1} - CG^{n+2} \tag{32}$$

which is an Abel equation, which may be integrated for some values of n.

(ii) *For* $v \neq 0$, $n = -v^2/(pv + v + 1)$. Equation (29) has a solution of the form

$$F = s^{1/n}. \tag{33}$$

In this case, if $p = m$, i.e. $v = 2$, $n = -4/(3+2p)$, $p = 0, 1, 2, \ldots$, the corresponding solutions of equation (1) follow from

$$u(x, t) = (x^2/t)^{-(3+2p)/4}.$$

In the case $p \neq m$, $m = 2$, $p = 1$, then $n = -\frac{1}{3}$ and equation (1) has the solution $u(x, t) = (x/t)^{-3}$, and for $m = 2$, $p = -1$, equation (1) has the solution $u(x, t) = xt$.

3. Discussion

We have attempted to find a comprehensive analytical solution to an inhomogeneous nonlinear diffusion equation (1) by applying the Lie similarity method. Furthermore, by applying the adjoint representation, the appropriate four optimal systems of similarity reductions are determined, which enables us to obtain a great variety of solutions. Our precedure extends some earlier results of some real physical processes. Because of the large number of different cases, it is not advisable to treat every case. Hence, we will consider three illustrative examples for some particular values of the parameter space (p, m) and n. Equation (1) has a wide range of applications in the case $m = p = \lambda$, where $\lambda\,(=0, 1, 2)$ is the space dimension, for both $n < 0$ ('fast' diffusion) and $n > 0$ ('slow' diffusion) [5, 6] and $n = 0$. Now we discuss briefly the solutions for some particular values of n and λ.

3.1. *Fast diffusion case*

We considered before the cases $n = -\frac{1}{2}, -1, -2, -2/p+1$. Let us now consider the case $p = m = 0$ and $n = -\frac{1}{2}$ which arises in models for plasma diffusion [13] and for the thermal expulsion of liquid helium [2, 14]. It has been noted in [15, 16] that equation (1) possesses a solution of the form $u = (a_0(x) + a_1(x)t)^2$. The typical behaviour of this solution can be obtained by equation (27) in a simple way, and the nature of the functions a_0, a_1 are determined. Another type of solution, in terms of Bessel's function, can be obtained by equations (15)–(17). It is important to emphasize that equation (1) has been reduced to ordinary differential equations, like equations (22) and (29), which can be solved numerically.

3.2. *Slow diffusion case*

Three cases of considerable interest will be discussed briefly:

(i) $\lambda = 0$, and $n > 0$, where equation (1) determines the transient temperature distribution and was obtained and studied in [17]. In the light of our results, equation (9) may be rewritten in terms of the new constants s_1 and R with relation (5) as

$$u(s) = Rt^{-k/2}(s_1^2 - s^2)^{1/n} \qquad s_1 > s > 0$$

which may be written in dimensionless form as

$$u(s)/u(0)=(1-(s/s_1)^2)^{1/n}$$

which agrees with that obtained in [17]. Other types of solution can be obtained by the similarity representations (24) and (27) as

$$u(x, t)=Rx^{2/n}(t_0-t)^{-1/n} \qquad \text{for } t_0>t$$

where t_0 is a new constant and $R=(Cn/4+2n)^{-1/n}$. With the similarity transformations (21) and (28), equation (1) can be reduced to ordinary differential equations like formulae (22) and (29), which can be solved numerically. An equation similar to expression (1) also appears for the pressure distribution in polytrophic gas flow through a porous medium.

(ii) $\lambda=0$, and $n=2$, where equation (1) represents a process of melting and evaporation of metals [18]. By using the similarity method we have the following types of solution: by relations (5) and (9)

$$u=t^{-1/4}\,(C-\tfrac{1}{4}x^2t^{-1/2})^{1/2}$$

and by relations (24) and (27),

$$u(x, t)=\tfrac{1}{2}x(t_0-t)^{-1/2} \qquad \text{for } t_0>t.$$

The similarity transformations (21) and (28) reduce equation (1) to an ordinary differential equation which can be solved numerically.

(iii) $\lambda=0$, and $n=1$, where equation (1) arises in other physical phenomena besides heat or chemical diffusion, for example the isothermal percolation of a perfect gas through a microporous medium [2]. A variety of solutions can be presented, as follows: by relations (5) and (9)

$$u=t^{-1/3}(C-\tfrac{1}{6}x^2t^{-2/3})$$

and by relations (24) and (27)

$$u(x, t)=x^2(C-6t)^{-1}.$$

The transformations (24) and (22) will reduce equation (1) to two ordinary differential equations like $FF_{SS}+F_S^2+kF-(k/2)SF_S=0$, where k is an arbitrary constant, and $4SFF_{SS}+4SF_S^2+2FF_S+SF_S=0$, which can be solved numerically.

3.3. For the case $n=0$, and $p=m=\lambda$, $\lambda=0, 1, 2$

In this case the one-dimensional heat conduction equation without the heat generation term can be represented by equation (1) for the linear, cylindrical or spherical coordinate system.

The fundamental solution of the linear diffusion equation ($\lambda=0$) is recovered by equation (11), where

$$u(x, t)=Ct^{-k/2}\exp(-kx^2/4t^k)$$

and k is an arbitrary constant. For $\lambda=1$, it has the solution

$$u(x, t)=Ct^{-1}\exp(-x^2/4t)$$

and for $\lambda=2$, it has the solution

$$u(x,t)=Ct^{-3/2}\exp(-x^2/4t).$$

The similarity transformation (21) reduces equation (1) for $\lambda=1,2$ to

$$SF_{SS}+\lambda F_S+kSF=0$$

where k is an arbitrary constant. Substituting $F(S)=S^{(1-\lambda)/2}g(Z)$, and $Z=k^{-1/2}S$, gives

$$Z^2g_{zz}+zg_z+(Z^2-b^2)g=0$$

where $b=(1-\lambda)/2$, and we get a solution of a Bessel function of order b [12].

Other types of solutions can be obtained by using the similarity transformation (28), where the reduced equation of relation (1) reads

$$4sF_{ss}+(a+s)F_s=0$$

where $a=2(1+\lambda)$ and $\lambda=1,2$, which has the solution for $\lambda=1$

$$u=F(s)=\ln s-\frac{s}{4\cdot 1!}+\frac{s^2}{4^2\cdot 2!}-\frac{s^3}{4^3\cdot 3!}+\ldots$$

and for $\lambda=2$ the solution

$$u=F(s)=2s^{-1/2}\,e^{-s/4}-\tfrac{1}{2}\int s^{-1/2}\,e^{-s/4}\,ds.$$

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